

# HOMOTOPY THEORY AND GENERALIZED DIMENSION SUBGROUPS

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**ABSTRACT.** Let  $G$  be a group and  $R, S, T$  its normal subgroups. There is a natural extension of the concept of commutator subgroup for the case of three subgroups  $\|R, S, T\|$  as well as the natural extension of the symmetric product  $\|\mathbf{r}, \mathbf{s}, \mathbf{t}\|$  for corresponding ideals  $\mathbf{r}, \mathbf{s}, \mathbf{t}$  in the integral group ring  $\mathbb{Z}[G]$ . In this paper, it is shown that the generalized dimension subgroup  $G \cap (1 + \|\mathbf{r}, \mathbf{s}, \mathbf{t}\|)$  has exponent 2 modulo  $\|R, S, T\|$ . The proof essentially uses homotopy theory. The considered generalized dimension quotient of exponent 2 is identified with a subgroup of the kernel of the Hurewicz homomorphism for the loop space over a homotopy colimit of classifying spaces.

## 1. INTRODUCTION

Let  $G$  be a group and  $\mathbb{Z}[G]$  its integral group ring. Every two-sided ideal  $\mathbf{a}$  in the integral group ring  $\mathbb{Z}[G]$  of a group  $G$  determines a normal subgroup

$$D(G, \mathbf{a}) := G \cap (1 + \mathbf{a})$$

of  $G$ . Such subgroups are called *generalized dimension subgroups*. The identification of generalized dimension subgroups is a fundamental problem in the theory of group rings. In general, given an ideal  $\mathbf{a}$ , the identification of  $D(G, \mathbf{a})$  is very difficult, for a survey on the problems in this area see [10], [14].

The idea that *the generalized dimension subgroups are related to the kernels of Hurewicz homomorphisms of certain spaces* was discussed in [14], [15], however, in the cited sources, all application of homotopical methods to the problems of group rings were related to very special cases. In this paper, we apply homotopy theory for a purely group-theoretical result of a more general type, namely to the description of the exponent of generalized dimension quotient constructed for a triple of normal subgroups in any group  $G$ .

Let  $G$  be a group and  $R, S$  its normal subgroups. Denote  $\mathbf{r} = (R - 1)\mathbb{Z}[G]$ ,  $\mathbf{s} = (S - 1)\mathbb{Z}[G]$ . It is proved in [2] that

$$(1) \quad D(G, \mathbf{rs} + \mathbf{sr}) = [R, S].$$

The following question arises naturally: how one can generalize the result (1) to the case of three and more normal subgroups of  $G$ . Our main result is the following.

**Theorem 1.** *Let  $G$  be a group and  $R, S, T$  its normal subgroups. Denote*

$$\mathbf{r} = (R - 1)\mathbb{Z}[G], \mathbf{s} = (S - 1)\mathbb{Z}[G], \mathbf{t} = (T - 1)\mathbb{Z}[G]$$

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and

$$\begin{aligned} \|R, S, T\| &:= [R, S \cap T][S, R \cap T][T, R \cap S] \\ \|\mathbf{r}, \mathbf{s}, \mathbf{t}\| &:= \mathbf{r}(\mathbf{s} \cap \mathbf{t}) + (\mathbf{s} \cap \mathbf{t})\mathbf{r} + \mathbf{s}(\mathbf{r} \cap \mathbf{t}) + (\mathbf{r} \cap \mathbf{t})\mathbf{s} + \mathbf{t}(\mathbf{r} \cap \mathbf{s}) + (\mathbf{r} \cap \mathbf{s})\mathbf{t}. \end{aligned}$$

Then, for every  $g \in D(G, \|\mathbf{r}, \mathbf{s}, \mathbf{t}\|)$ ,  $g^2 \in \|R, S, T\|$ , i.e. the generalized dimension quotient

$$(2) \quad \frac{D(G, \|\mathbf{r}, \mathbf{s}, \mathbf{t}\|)}{\|R, S, T\|}$$

is a  $\mathbb{Z}/2$ -vector space.

The proof of theorem 1 consists of the following steps. First we show that there exists a space  $X$  such that there is a commutative diagram

$$(3) \quad \begin{array}{ccc} \frac{R \cap S \cap T}{\|R, S, T\|} & \longrightarrow & \frac{\mathbf{r} \cap \mathbf{s} \cap \mathbf{t}}{\|\mathbf{r}, \mathbf{s}, \mathbf{t}\|} \\ \parallel & & \downarrow \\ \pi_2(\Omega X) & \xrightarrow{h_2 \Omega} & H_2(\Omega X) \end{array}$$

where the lower horizontal map is the Hurewicz homomorphism. Secondly, we show that, for any space  $X$ , the kernel of the Hurewicz homomorphism

$$\Omega h_2 : \pi_2(\Omega X) \rightarrow H_2(\Omega X)$$

is a 2-torsion subgroup of  $\pi_2(\Omega X) = \pi_3(X)$ .

There are examples of groups with triples of subgroups such that the generalized dimension quotient 2 is non-trivial. Let  $F = F(a, b, c)$  be a free group with basis  $\{a, b, c\}$ . Consider the following normal subgroups of  $F$ :

$$R = \langle a^2, c \rangle^F, \quad S = \langle a, bc^{-1} \rangle^F, \quad T = \langle a, b \rangle^F.$$

Then, there exists the following natural commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longrightarrow & \frac{R \cap S \cap T}{\|R, S, T\|} & \longrightarrow & \frac{\mathbf{r} \cap \mathbf{s} \cap \mathbf{t}}{\|\mathbf{r}, \mathbf{s}, \mathbf{t}\|} \\ \parallel & & \parallel & & \parallel \\ \mathbb{Z}/2 & \longrightarrow & \pi_2(\Omega \Sigma \mathbb{R} P^2) & \xrightarrow{h_2 \Omega} & H_2(\Omega \Sigma \mathbb{R} P^2) \end{array}$$

This example and discussion of a generalization of the considered construction to the case of  $> 3$  normal subgroups is given in section 5.

Another application of homotopic methods is the following identification of the generalized dimension subgroup (see theorem 9):

$$(4) \quad D(G, \mathbf{r}\mathbf{s} + \mathbf{s}\mathbf{r} + (\mathbf{r} \cap \mathbf{s})\mathbf{t} + \mathbf{t}(\mathbf{r} \cap \mathbf{s})) = [R, S][R \cap S, T].$$

This generalizes (1), indeed, (1) is equivalent to (4) for  $T = 1$ .

The space  $X$  from (3) is the homotopy colimit of the cubic diagram of eight classifying spaces  $BG, B(G/R), B(G/S), B(G/T), B(G/RS), B(G/RT), B(G/ST)$ . The left vertical isomorphism in (3) is proved in [7]. In section 3 we develop the theory of cubes of fibrations in the category of simplicial non-unital rings and correspondence between  $n$ -cubes of fibrations with crossed  $n$ -cubes of rings. We

obtain ring-theoretical analogs of the result from [8]. Note that, in this paper, we do not consider the properties of universality of crossed  $n$ -cubes of rings. The universality property is needed for an explicit description of the homology groups  $H_*(\Omega X)$  of homotopy colimits  $X$  of classifying spaces (see the proof of theorem 1 in [7]). For the reason of this paper, namely, for an analysis of generalized dimension subgroups, only crossed properties of the diagrams of rings are enough and these properties are given in section 3.

## 2. HUREWICZ HOMOMORPHISM

**2.1. Two lemmas about squares of abelian groups.** First we state two lemmas about squares of abelian groups. These lemmas are advanced versions of well known statements (see [9, Part 1, 6.2.6]). We give them without a prove because it is standard. Authors learned these lemmas from non-formal discussions with Alexander Generalov.

Consider a square of abelian groups  $\mathcal{S}$  with induced homomorphisms on kernels and cokernels:

$$\begin{array}{ccccccc}
 & & \text{Ker}(g) & \xrightarrow{\tilde{f}} & \text{Ker}(g') & & \\
 & & \downarrow & & \downarrow & & \\
 \text{Ker}(f) & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \text{Coker}(f) \\
 \downarrow \tilde{g} & & \downarrow g & & \downarrow g' & & \downarrow \tilde{g}' \\
 \text{Ker}(f') & \longrightarrow & C & \xrightarrow{f'} & D & \longrightarrow & \text{Coker}(f') \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Coker}(g) & \xrightarrow{\tilde{f}'} & \text{Coker}(g') & & 
 \end{array}$$

$\mathcal{S}$

and maps  $A \rightarrow B \oplus C \rightarrow D$  given by  $a \mapsto (f(a), -g(a))$  and  $(b, c) \mapsto g'(b) + f'(c)$ .

**Lemma 2.** *The following statements are equivalent.*

- (1)  $\mathcal{S}$  is a pushout square.
- (2) The sequence  $A \rightarrow B \oplus C \rightarrow D \rightarrow 0$  is exact.
- (3)  $\tilde{g}'$  is an isomorphism and  $\tilde{g}$  is an epimorphism.
- (4)  $\tilde{f}'$  is an isomorphism and  $\tilde{f}$  is an epimorphism.

**Lemma 3.** *The following statements are equivalent.*

- (1)  $\mathcal{S}$  is a pullback square.
- (2) The sequence  $0 \rightarrow A \rightarrow B \oplus C \rightarrow D$  is exact.
- (3)  $\tilde{g}$  is an isomorphism and  $\tilde{g}'$  is a monomorphism.
- (4)  $\tilde{f}$  is an isomorphism and  $\tilde{f}'$  is a monomorphism.

**2.2. Whitehead quadratic functor.** For an abelian group  $A$ , the Whitehead group  $\Gamma(A)$  is generated by symbols  $\gamma(a)$ ,  $a \in A$  with the following relations

$$\gamma(0) = 0,$$

$$\gamma(-a) = \gamma(a), \quad a \in A$$

$$\gamma(a + b + c) - \gamma(a + b) - \gamma(a + c) - \gamma(b + c) + \gamma(a) + \gamma(b) + \gamma(c) = 0, \quad a, b, c \in A.$$

The correspondence  $A \mapsto \Gamma(A)$  defines a quadratic functor in the category of abelian groups called the *Whitehead quadratic functor*. It has the following simple properties

$$\begin{aligned}\Gamma(\mathbb{Z}/n) &= \mathbb{Z}/(2n, n^2) \\ A \otimes B &= \text{Ker}\{\Gamma(A \oplus B) \rightarrow \Gamma(A) \oplus \Gamma(B)\}.\end{aligned}$$

There is a natural transformation of functors  $\Gamma \rightarrow \otimes^2$  defined, for an abelian group  $A$ , as  $\gamma(a) \mapsto a \otimes a$ ,  $a \in A$ .

Define the functor  $\Phi$  as a kernel of  $\Gamma \rightarrow \otimes^2$ . Then, for any abelian group  $A$ , there is a natural exact sequence

$$0 \rightarrow \Phi(A) \rightarrow \Gamma(A) \rightarrow A \otimes A \rightarrow \Lambda^2(A) \rightarrow 0$$

where  $\Lambda^2$  is the exterior square. One can easily check that, for any pair of abelian groups  $A, B$ , the (bi)natural map between the cross-effects of the functors  $\Gamma$  and  $\otimes^2$

$$\Gamma(A|B) = A \otimes B \rightarrow \otimes^2(A|B) = A \otimes B \oplus B \otimes A$$

is a monomorphism. From this property together with the above description of the values of  $\Gamma$  for cyclic groups follows that  $\Phi$  is a 2-torsion functor, i.e. *for any  $A$  and  $a \in \Phi(A)$ ,  $2a = 0$  in  $\Phi(A)$* .

### 2.3. Kernel of the Hurewicz homomorphism.

**Proposition 4.** *For any connected space  $X$ , the kernel of the Hurewicz homomorphism*

$$h_2\Omega : \pi_2(\Omega X) \rightarrow H_2(\Omega X)$$

*is a 2-torsion subgroup of  $\pi_2(\Omega X) = \pi_3(X)$ .*

Observe that, the statement about the third Hurewicz homomorphism obviously is not true without taking loops. For any odd prime  $p$ , the Moore space  $P^3(p)$  has  $\pi_3(P^3(p)) = \mathbb{Z}/p$  and  $H_3(P^3(p)) = 0$ .

*Proof.* First consider the case of a simply-connected space  $Y$ . Let  $GY$  be the simplicial Kan loop construction. The following diagram of fibrations

$$\begin{array}{ccccc} [GY, GY] & \twoheadrightarrow & GY & \twoheadrightarrow & (GY)_{ab} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^2(GY) & \twoheadrightarrow & \mathbb{Z}[GY] & \twoheadrightarrow & \mathbb{Z}[GY]/\Delta^2(GY) \end{array}$$

induces the commutative diagram of homotopy groups

$$(5) \quad \begin{array}{ccccccc} H_4(Y) & \longrightarrow & \pi_2([GY, GY]) & \longrightarrow & \pi_2(\Omega Y) & \twoheadrightarrow & H_3(Y) \\ \parallel & & \downarrow & & \downarrow h_2\Omega & & \parallel \\ H_4(Y) & \longrightarrow & \pi_2(\Delta^2(GY)) & \longrightarrow & H_2(\Omega Y) & \twoheadrightarrow & H_3(Y) \end{array}$$

A simple analysis of connectivity of the simplicial groups  $[GY, GY]$  and  $\Delta^2(GY)$  shows that there are natural isomorphisms

$$\begin{aligned}\pi_2([GY, GY]) &= \pi_2([GY, GY]/[[GY, GY], GY]) = \pi_2(\Lambda^2((GY)_{ab})), \\ \pi_2(\Delta^2(GY)) &= \pi_2(\Delta^2(GY)/\Delta^3(GY)) = \pi_2((GY)_{ab} \otimes (GY)_{ab}).\end{aligned}$$

The derived functors of  $\Lambda^2$  and  $\otimes^2$  are well-known in a general situation (see, for example, [1]). We obtain the following natural diagram

$$\begin{array}{ccc}\pi_2([GY, GY]) & \longrightarrow & \pi_2(\Delta^2(GY)) \\ \parallel & & \parallel \\ \Gamma(H_2(Y)) & \longrightarrow & H_2(Y) \otimes H_2(Y)\end{array}$$

The left hand isomorphism in the last diagram is a reformulation of the result due to Whitehead [17], the right hand isomorphism follows from the Kunneth formula. Now lemmas 2 and 3 imply that, the diagram (5) can be extended to the following diagram

$$\begin{array}{ccccccc}\Phi(H_2(Y)) & \longrightarrow & K & & & & \\ \downarrow & & \downarrow & & & & \\ H_4(Y) & \longrightarrow & \Gamma(H_2(Y)) & \longrightarrow & \pi_3(Y) & \longrightarrow & H_3(Y) \\ \parallel & & \downarrow & & \downarrow h_2\Omega & & \parallel \\ H_4(Y) & \longrightarrow & H_2(Y) \otimes H_2(Y) & \longrightarrow & H_2(\Omega Y) & \longrightarrow & H_3(Y) \\ & & \downarrow & & \downarrow & & \\ & & H_2(Y) \wedge H_2(Y) & \equiv & H_2(\pi_1(\Omega Y)) & & \end{array}$$

where the upper horizontal map is an epimorphism. Since the group  $\Phi(H_2(Y))$  is 2-torsion, the kernel  $K$  of the Hurewicz homomorphism also is 2-torsion and the needed statement is proved.

Now consider the case of arbitrary connected space  $X$ . Consider its universal cover  $\tilde{X} \rightarrow X$ . The needed statement follows from the diagram

$$\begin{array}{ccc}\pi_2(\Omega \tilde{X}) & \equiv & \pi_2(\Omega X) \\ \downarrow h_2\Omega & & \downarrow h_2\Omega \\ H_2(\Omega \tilde{X}) & \twoheadrightarrow & H_2(\Omega X)\end{array}$$

and the above proof of the statement for the simply-connected case.  $\square$

### 3. CUBES OF SIMPLICIAL NON-UNITAL RINGS AND THEIR CROSSED CUBES

**3.1. Cubes of fibrations and fibrant cubes.** Set  $\langle n \rangle = \{1, \dots, n\}$ . By a ring we assume a **non-unital** ring, and by a ring homomorphism we assume a non-unital ring homomorphism.

Consider the category of simplicial rings (s.r.) as a model category, whose weak equivalences are weak equivalences of underlying simplicial sets and fibrations are level-wise surjective homomorphisms (see Ch.2  $\Sigma 4$  [16]). Then a fibration sequence in  $\mathbf{sRng}$  is isomorphic to a sequence of the form

$$(6) \quad I \rightarrow R \twoheadrightarrow R/I,$$

where  $I$  is an ideal of the simplicial ring  $R$ .

Consider the ordered sets  $\{0, 1\}$  and  $\{-1, 0, 1\}$  as categories in a usual way. Let  $\mathcal{F} : \{-1, 0, 1\}^n \rightarrow \mathbf{sRng}$  be a functor. For two disjoint subsets  $\alpha, \beta \subseteq \langle n \rangle$  (i.e.  $\alpha \cap \beta = \emptyset$ ) we put

$$\mathcal{F}(\alpha, \beta) = \mathcal{F}(i_1, \dots, i_n),$$

where  $\alpha = \{k \mid i_k = 1\}$  and  $\beta = \{k \mid i_k = -1\}$ . Then, if  $\alpha' \supseteq \alpha$  and  $\beta' \subseteq \beta$ , we have a map

$$\mathcal{F}(\alpha, \beta) \longrightarrow \mathcal{F}(\alpha', \beta').$$

An  **$n$ -cube of fibrations** of s.r. (see [13, 1.3]) is a functor  $\mathcal{F} : \{-1, 0, 1\}^n \rightarrow \mathbf{sRng}$  such that for any disjoint subsets  $\alpha, \beta \subseteq \langle n \rangle$  and  $k \in \langle n \rangle \setminus (\alpha \cup \beta)$  we have a fibration sequence

$$\mathcal{F}(\alpha, \beta \cup \{k\}) \rightarrow \mathcal{F}(\alpha, \beta) \twoheadrightarrow \mathcal{F}(\alpha \cup \{k\}, \beta).$$

If  $n = 2$  it is a  $3 \times 3$  square whose rows and columns are fibration sequences:

$$\begin{array}{ccccc} \mathcal{F}(\emptyset, \langle 2 \rangle) & \longrightarrow & \mathcal{F}(\emptyset, \{2\}) & \longrightarrow & \mathcal{F}(\{1\}, \{2\}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(\emptyset, \{1\}) & \longrightarrow & \mathcal{F}(\emptyset, \emptyset) & \longrightarrow & \mathcal{F}(\{1\}, \emptyset) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(\{2\}, \{1\}) & \longrightarrow & \mathcal{F}(\{2\}, \emptyset) & \longrightarrow & \mathcal{F}(\langle 2 \rangle, \emptyset). \end{array}$$

An  **$n$ -cube** of s.r. is a functor  $\mathcal{R} : \{0, 1\}^n \rightarrow \mathbf{sRng}$ . We set

$$\mathcal{R}(\alpha) = \mathcal{R}(i_1, \dots, i_n),$$

where  $\alpha = \{k \mid i_k = 1\}$ . It is easy to see that  $\{0, 1\}^n$  is a direct category, and hence, it is a Reedy category. It follows that there is a natural model structure on the category of  $n$ -cubes of simplicial rings, called Reedy model structure [12], [11]. Weak equivalences of  $n$ -cubes in this model structure are defined level-wise. An  $n$ -cube  $\mathcal{R}$  is **fibrant** in this model structure if and only if the map

$$\mathcal{R}(\alpha) \longrightarrow \lim_{\alpha' \supsetneq \alpha} \mathcal{R}(\alpha')$$

is a fibration of s.r. for any  $\alpha \subseteq \langle n \rangle$ .

Let  $\mathcal{R}$  be an  $n$ -cube of s.r. We consider the functor  $\mathbf{E}_{\text{fib}}(\mathcal{R}) : \{-1, 0, 1\}^n \rightarrow \mathbf{sRng}$  given by

$$(7) \quad \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta) = \text{Ker} \left( \mathcal{R}(\alpha) \rightarrow \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i\}) \right)$$

with obvious morphisms.

**Lemma 5.** *Let  $\mathcal{R}$  be an  $n$ -cube of s.r. Then the following statements are equivalent.*

- (1)  $\mathcal{R}$  is fibrant.
- (2)  $\mathcal{R}$  can be embedded into an  $n$ -cube of fibrations.
- (3)  $\mathbf{E}_{\text{fib}}(\mathcal{R})$  is an  $n$ -cube of fibrations.

Moreover, if  $\mathcal{R}$  is a fibrant  $n$ -cube of s.r.,  $\mathbf{E}_{\text{fib}}(\mathcal{R})$  is the unique (up to unique isomorphism that respects the embeddings)  $n$ -cube of fibrations to which  $\mathcal{R}$  can be embedded.

*Proof.* If  $d \geq 0$ ,  $k \notin \alpha$  and  $r \in \mathcal{R}(\alpha)_d$ , we denote by  $r^k$  the image of  $r$  in  $\mathcal{R}(\alpha \cup \{k\})_d$ . Assume that  $\alpha, \beta \subseteq \langle n \rangle$  are disjoint sets and  $d \geq 0$ . An  $(\alpha, \beta)$ -**collection** is a collection  $(r_i) \in \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i\})$  such that  $r_i^j = r_j^i$  for any  $i, j \in \beta$ . A **lifting** of an  $(\alpha, \beta)$ -collection  $(r_i)$  is an element  $r \in \mathcal{R}(\alpha)_d$  such that  $r_i = r^i$ . It is easy to see that the ring  $\lim_{\alpha' \supseteq \alpha} \mathcal{R}(\alpha')_d$  consist of  $(\alpha, \alpha^c)$ -collections, where  $\alpha^c = \langle n \rangle \setminus \alpha$ . Then  $\mathcal{R}$  is fibrant if and only if for any  $(\alpha, \alpha^c)$ -collection there exists a lifting. We claim that if  $\mathcal{R}$  is fibrant then for any disjoint  $\alpha, \beta$  and any  $(\alpha, \beta)$ -collection there exist a lifting. The prove is by induction on  $|\langle n \rangle \setminus (\alpha \cup \beta)|$ . If it is equal to 0, we done. Assume that  $(r_i)$  is an  $(\alpha, \beta)$ -collection. Consider any  $j \in \langle n \rangle \setminus (\alpha \cup \beta)$  and the  $(\alpha \cup \{j\}, \beta)$ -collection  $(r_i^j)$ . By induction hypothesis we have a lifting  $r_j \in \mathcal{R}(\alpha \cup \{j\})_d$  of  $(r_i^j)$ . Then we get a  $(\alpha, \alpha^c)$ -collection  $(r_i)_{i \in \alpha^c}$ , whose lifting is the lifting of the original  $(\alpha, \beta)$ -collection  $(r_i)$ . Therefore  $\mathcal{R}$  is fibrant if and only if any  $(\alpha, \beta)$ -collection has a lifting.

(1) $\Rightarrow$ (3). Assume that  $\mathcal{R}$  is fibrant and prove that  $\mathbf{E}_{\text{fib}}(\mathcal{R})$  is an  $n$ -cube of fibrations. Consider the diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta \cup \{k\}) & \longrightarrow & \mathcal{R}(\alpha) & \longrightarrow & \prod_{i \in \beta \cup \{k\}} \mathcal{R}(\alpha \cup \{i\}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta) & \longrightarrow & \mathcal{R}(\alpha) & \longrightarrow & \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i\}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta) & \longrightarrow & \mathcal{R}(\alpha \cup \{k\}) & \longrightarrow & \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i, k\}).
 \end{array}$$

We have to prove that the left column is a short exact sequence. The only non-obvious thing is that  $\mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta)_d \rightarrow \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta)_d$  is surjective. Consider any  $r_k \in \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta)_d$  and denote  $r_i = 0$  for  $i \in \beta$ . Then  $(r_i)$  is a  $(\alpha, \beta \cup \{k\})$ -collection, whose lifting is a preimage of  $r_k$ .

(3) $\Rightarrow$ (1). Assume that  $\mathbf{E}_{\text{fib}}(\mathcal{R})$  is an  $n$ -cube of fibrations and prove that  $\mathcal{R}$  is fibrant. We need to prove that for any  $(\alpha, \beta)$ -collection there exists a lifting. Prove it by induction on  $|\beta|$ . If  $\beta = \emptyset$ , it is obvious. Assume that it holds for  $\beta$  and prove it for  $\beta' = \beta \cup \{k\}$ , where  $k \in \langle n \rangle \setminus (\alpha \cup \beta)$ . Consider an  $(\alpha, \beta \cup \{k\})$ -collection  $(r_i)_{i \in \beta \cup \{k\}}$ . By induction hypotheses its  $(\alpha, \beta)$ -subcollection  $(r_i)_{i \in \beta}$  has a lifting  $\bar{r} \in \mathcal{R}(\alpha)_d$ . Then  $r_k - \bar{r}^k \in \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta)_d$ . Since the map  $\mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta)_d \rightarrow \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta)_d$  is surjective we get a preimage  $\hat{r} \in \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta)_d$  such that  $\hat{r}^k = r_k - \bar{r}^k$ . Then  $r = \hat{r} + \bar{r}$  is a lifting of the  $(\alpha, \beta \cup \{k\})$ -collection  $(r_i)_{i \in \beta \cup \{k\}}$ .

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Assume that  $\mathcal{R}$  is embedded into an  $n$ -cube of fibrations  $\mathcal{F}$ . Replacing  $\mathcal{F}$  by isomorphic one, we can assume that the fibres are identical embeddings. Prove

that  $\mathcal{F}(\alpha, \beta) = \mathbf{E}_{\text{fib}}(\alpha, \beta)$  by induction on  $|\beta|$ . If  $\beta = \emptyset$ , then  $\mathcal{F}(\alpha, \emptyset) = \mathcal{R}(\alpha) = \mathbf{E}_{\text{fib}}(\alpha, \emptyset)$ . Assume that it holds for  $\beta$  and prove it for  $\beta' = \alpha \cup \{k\}$ . By induction hypothesis we have a commutative diagram

$$\begin{array}{ccccc}
 & 0 & & & \\
 & \downarrow & & & \\
 & \mathcal{F}(\alpha, \beta \cup \{k\}) & \longrightarrow & \mathcal{R}(\alpha) & \longrightarrow & \prod_{i \in \beta \cup \{k\}} \mathcal{R}(\alpha \cup \{i\}) \\
 & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta) & \longrightarrow & \mathcal{R}(\alpha) & \longrightarrow & \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i\}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha \cup \{k\}, \beta) & \longrightarrow & \mathcal{R}(\alpha \cup \{k\}) & \longrightarrow & \prod_{i \in \beta} \mathcal{R}(\alpha \cup \{i, k\}),
 \end{array}$$

where the right column is a fibration sequence. Using that  $\mathcal{F}(\alpha, \beta \cup \{k\})_d = \text{Ker}(\mathbf{E}_{\text{fib}}(\alpha, \beta)_d \rightarrow \mathbf{E}_{\text{fib}}(\alpha \cup \{k\}, \beta)_d)$  it is easy to deduce from the diagram that  $\mathcal{F}(\alpha, \beta \cup \{k\})_d = \text{Ker}(\mathcal{R}(\alpha)_d \rightarrow \prod_{i \in \beta \cup \{k\}} \mathcal{R}(\alpha \cup \{i\})_d) = \mathbf{E}_{\text{fib}}(\mathcal{R})(\alpha, \beta \cup \{k\})_d$ .  $\square$

**3.2. Cubes of fibrations and good tuples of ideals.** An  $n$ -tuple of ideals of a s.r. is a tuple  $I = (R; I_1, \dots, I_n)$ , where  $R$  is a simplicial ring and  $I_i$  are (simplicial) ideals of  $R$ . For  $\beta \subseteq \langle n \rangle$  we set

$$I(\beta) = \bigcap_{i \in \beta} I_i.$$

An  $n$ -tuple of ideals  $I$  is said to be **good** if for any disjoint subsets  $\alpha, \beta \subset \langle n \rangle$  and  $k \in \langle n \rangle \setminus (\alpha \cup \beta)$  the following equality holds

$$(8) \quad I(\beta \cup \{k\}) \cap \left( \sum_{i \in \alpha} I(\beta \cup \{i\}) \right) = \sum_{i \in \alpha} I(\beta \cup \{k, i\}).$$

It is easy to check that any 2-tuple of ideals is always good. But a 3-tuple of ideals is good if and only if for any  $i, j, k \in \{1, 2, 3\}$  the following holds  $I_i \cap (I_j + I_k) = I_i \cap I_j + I_i \cap I_k$ .

For an  $n$ -tuple of ideals  $I = (R; I_1, \dots, I_n)$  we consider the functor  $\mathbf{E}_{\text{idl}}(I) : \{-1, 0, 1\}^n \rightarrow \mathbf{sRng}$  given by

$$(9) \quad \mathbf{E}_{\text{idl}}(I)(\alpha, \beta) = \frac{I(\beta)}{\sum_{i \in \alpha} I(\beta \cup \{i\})}$$

with obvious morphisms. For example  $\mathbf{E}_{\text{idl}}(R; I, J)$  looks as follows:

$$\begin{array}{ccccc}
 I \cap J & \longrightarrow & I & \longrightarrow & I/(I \cap J) \\
 \downarrow & & \downarrow & & \downarrow \\
 J & \longrightarrow & R & \longrightarrow & R/J \\
 \downarrow & & \downarrow & & \downarrow \\
 J/(I \cap J) & \longrightarrow & R/I & \longrightarrow & R/(I + J).
 \end{array}$$



Note that this definition can be rewritten in a way dual to (7):  $\mathbf{E}_{\text{idl}}(I)(\alpha, \beta) = \text{Coker}(\prod_{i \in \alpha} I(\beta \cup \{i\}) \rightarrow I(\beta))$ .

For an  $n$ -cube of fibrations  $\mathcal{F}$  we consider an  $n$ -tuple of ideals

$$(10) \quad \mathbf{T}_{\text{idl}}(\mathcal{F}) = (R; I_1, \dots, I_n), \quad R = \mathcal{F}(\emptyset, \emptyset), \quad I_i = \text{Ker}(\mathcal{F}(\emptyset, \emptyset) \rightarrow \mathcal{F}(\{i\}, \emptyset)).$$

**Lemma 6.** *Let  $I$  be an  $n$ -tuple of ideals of a s.r. Then the following statements are equivalent.*

- (1)  $I$  is good;
- (2)  $I = \mathbf{T}_{\text{idl}}(\mathcal{F})$  for some  $n$ -cube of fibrations  $\mathcal{F}$ ;
- (3)  $\mathbf{E}_{\text{idl}}(I)$  is an  $n$ -cube of fibrations.

Moreover, if  $I$  is good,  $\mathbf{E}_{\text{idl}}(I)$  is the unique (up to unique isomorphism that respects the equalities)  $n$ -cube of fibrations such that  $I = \mathbf{T}_{\text{idl}}(\mathbf{E}_{\text{idl}}(I))$ .

*Proof.* (1) $\Leftrightarrow$ (3). Let  $\alpha, \beta \subset \langle n \rangle$  be disjoint sets and  $k \in \langle n \rangle \setminus (\alpha \cup \beta)$ . Consider the sequence

$$\frac{I(\beta \cup \{k\})}{\sum_{i \in \alpha} I(\beta \cup \{i, k\})} \longrightarrow \frac{I(\beta)}{\sum_{i \in \alpha} I(\beta \cup \{i\})} \longrightarrow \frac{I(\beta)}{\sum_{i \in \alpha \cup \{k\}} I(\beta \cup \{i\})}.$$

It is easy to see that the right hand map is an epimorphism and that it is exact in the middle term. Moreover, the left hand homomorphism is a monomorphism if and only if (8) holds.

(3) $\Rightarrow$ (2). Follows from the equality  $\mathbf{T}_{\text{idl}}(\mathbf{E}_{\text{idl}}(I)) = I$ .

(2) $\Rightarrow$ (3). Assume that  $I = \mathbf{T}_{\text{idl}}(\mathcal{F})$  for some  $n$ -cube of fibrations  $\mathcal{F}$ . Replacing  $\mathcal{F}$  by isomorphic one, we can assume that the fibres are identical embeddings. First we prove that  $I(\beta) = \mathcal{F}(\emptyset, \beta)$ . The prove is by induction on  $|\beta|$ . If  $|\beta| = 0, 1$  it is obvious. Assume that  $|\beta| \geq 2$  and fix two distinct elements  $k, l \in \beta$ . By induction hypothesis we have  $\mathcal{F}(\emptyset, \beta \setminus \{k\}) = I(\beta \setminus \{k\})$  and  $\mathcal{F}(\emptyset, \beta \setminus \{l\}) = I(\beta \setminus \{l\})$ . Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(\emptyset, \beta) & \longrightarrow & I(\beta \setminus \{k\}) & \longrightarrow & \mathcal{F}(\{k\}, \beta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I(\beta \setminus \{l\}) & \longrightarrow & I(\beta \setminus \{k, l\}) & \longrightarrow & \mathcal{F}(\{k\}, \beta \setminus \{k, l\}) \longrightarrow 0, \end{array}$$

whose rows are short exact sequences. Since the left square consists of monomorphisms and the map on the cokernels  $\mathcal{F}(\{k\}, \beta) \rightarrow \mathcal{F}(\{k\}, \beta \setminus \{k, l\})$  is a monomorphism, by Lemma 3 we get that the left square is a pullback square. Hence,  $\mathcal{F}(\emptyset, \beta) = I(\beta \setminus \{k\}) \cap I(\beta \setminus \{l\}) = I(\beta)$ .

So we have  $\mathcal{F}(\emptyset, \beta) = \mathbf{E}_{\text{idl}}(\emptyset, \beta)$ . Further we prove by induction on  $|\alpha|$  that there is a unique isomorphism  $\mathcal{F}(\alpha, \beta) \cong \mathbf{E}_{\text{idl}}(I)(\alpha, \beta)$  that satisfies commutation properties and lifts this equality for  $\alpha = \emptyset$ . Assume that this holds for  $\alpha$  and prove it for  $\alpha \cup \{k\}$ . By the induction hypothesis we have that  $\mathcal{F}(\alpha, \beta \cup \{k\}) = \frac{I(\beta \cup \{k\})}{\sum_{i \in \alpha} I(\beta \cup \{i, k\})}$  and  $\mathcal{F}(\alpha, \beta) = \frac{I(\beta)}{\sum_{i \in \alpha} I(\beta \cup \{i\})}$ . It follows that there is a short exact sequence

$$0 \rightarrow \frac{I(\beta \cup \{k\})}{\sum_{i \in \alpha} I(\beta \cup \{i, k\})} \longrightarrow \frac{I(\beta)}{\sum_{i \in \alpha} I(\beta \cup \{i\})} \longrightarrow \mathcal{F}(\alpha \cup \{k\}, \beta) \rightarrow 0,$$

which induces the required isomorphism  $\mathcal{F}(\alpha, \beta) \cong \frac{I(\beta)}{\sum_{i \in \alpha \cup \{k\}} I(\beta \cup \{i\})}$ .  $\square$

**3.3. Three equivalent categories.** Consider the truncation functor

$$(11) \quad T_{\text{fib}} : (n\text{-cubes of fibrations of s.r.}) \longrightarrow (\text{fibrant } n\text{-cubes of s.r.})$$

that induced by the embedding  $\{0, 1\}^n \subset \{-1, 0, 1\}^n$ . Lemma 5 implies that this functor is well defined.

**Proposition 7.** *The functors*

$$(12) \quad \begin{array}{ccc} & (\text{fibrant } n\text{-cubes of s.r.}) & \\ & \uparrow T_{\text{fib}} \quad \sim \quad \downarrow E_{\text{fib}} & \\ & (n\text{-cubes of fibrations of s.r.}) & \\ & \uparrow E_{\text{idl}} \quad \sim \quad \downarrow T_{\text{idl}} & \\ & (\text{good } n\text{-tuples of ideals of s.r.}) & \end{array}$$

given by (7), (9), (11) and (10) define mutually invert equivalences of categories.

*Proof.* The equalities  $T_{\text{fib}}E_{\text{fib}} = \text{Id}$ ,  $T_{\text{idl}}E_{\text{idl}} = \text{Id}$  are obvious. The isomorphisms  $E_{\text{fib}}T_{\text{idl}} \cong \text{Id}$  and  $E_{\text{idl}}T_{\text{fib}} \cong \text{Id}$  follow from Lemma 5 and Lemma 6.  $\square$

Consider the functor

$$(13) \quad \text{Fibre} : (\text{fibrant } n\text{-cubes of s.r.}) \xrightarrow{\sim} (\text{good } n\text{-tuples of ideals of s.r.}),$$

given by  $\text{Fibre}(\mathcal{R}) = (R; I_1, \dots, I_n)$ , where  $R = \mathcal{R}(\emptyset)$  and  $I_i = \text{Ker}(\mathcal{R}(\emptyset) \rightarrow \mathcal{R}(\{i\}))$ .

**Corollary 8.** *The functor (13) is an equivalence of categories.*

**3.4. Crossed cubes of cubes of simplicial rings.** Following Ellis [6] we define a **crossed  $n$ -cube of rings**  $\{R_\beta\}$  as a family of rings, where  $\beta \subseteq \langle n \rangle$  together with homomorphisms  $\mu_i : R_\beta \rightarrow R_{\beta \setminus \{i\}}$  and  $h : R_\beta \otimes R_{\beta'} \rightarrow R_{\beta \cup \beta'}$  such that for  $a, a' \in R_\beta$ ,  $b, b' \in R_{\beta'}$ ,  $c \in R_{\beta''}$  and  $i, j \in \langle n \rangle$  such that

- $\mu_i a = a$  if  $i \notin \beta$ ;
- $\mu_i \mu_j a = \mu_j \mu_i a$ ;
- $\mu_i h(a \otimes b) = h(\mu_i a \otimes b) = h(a \otimes \mu_i b)$ ;
- $h(a \otimes b) = h(\mu_i a \otimes b) = h(a \otimes \mu_i b)$  if  $i \in \beta \cap \beta'$ ;
- $h(a \otimes a') = aa'$ ;

with the associative property:

- $h(h(a \otimes b) \otimes c) = h(a \otimes h(b \otimes c))$ .

Morphisms of crossed  $n$ -cubes are defined obviously. Consider the functor

$$(14) \quad \pi_0 : (n\text{-tuples of ideals of s.r.}) \longrightarrow (\text{crossed } n\text{-cubes of r.})$$

that sends  $I$  to  $\{R_\beta\}$ , where  $R_\beta = \pi_0 I(\beta)$ ,  $\mu_i = \pi_0(I(\beta) \hookrightarrow I(\beta \setminus \{i\}))$  and  $h : R_\beta \otimes R_{\beta'} \rightarrow R_{\beta \cup \beta'}$  is the composition of the isomorphism  $\pi_0 I(\beta) \otimes \pi_0 I(\beta') \cong \pi_0(I(\beta) \otimes I(\beta'))$  and the map  $\pi_0(I(\beta) \otimes I(\beta') \rightarrow I_{\beta \cup \beta'})$ . It is easy to check that  $\{R_\beta\}$  is a crossed  $n$ -cube of rings.

In the Reedy model structure on the category of  $n$ -cubes there is a functorial fibrant replacement

$$\gamma : \mathcal{R} \xrightarrow{\sim} \overline{\mathcal{R}},$$

where  $\overline{\mathcal{R}}$  is a fibrant  $n$ -cube and  $\gamma$  is a weak equivalence and a cofibration. Then consider the functor

$$(15) \quad \Pi : (n\text{-cubes of s.r.}) \longrightarrow (\text{crossed } n\text{-cubes of rings}),$$

given by  $\Pi(\mathcal{R}) = \pi_0(\text{Fibre}(\overline{\mathcal{R}}))$ , which is analogue of the one constructed in [3] for simplicial rings. Analysing the definition of  $\Pi$  we get the following. If  $\mathcal{R}$  is an  $n$ -cube of s.r. we can embed it into a  $n$ -cube of **homotopy** fibration sequences  $\mathcal{F}$  in the **homotopy category** of simplicial rings by taking homotopy fibres of all arrows, and then

$$\Pi(\mathcal{R})_\beta = \pi_0(\mathcal{F}(\emptyset, \beta)).$$

#### 4. PROOF OF THEOREM 1

**4.1. Proof of theorem 1.** Now suppose that  $R, S, T$  are normal subgroups of a group  $G$ . Let  $X$  be a homotopy pushout of the following diagram of classifying spaces:

$$(16) \quad \begin{array}{ccccc} & BG & \xrightarrow{\quad} & B(G/R) & \\ & \swarrow & & \searrow & \\ B(G/S) & \xrightarrow{\quad} & B(G/RS) & & \\ \downarrow & & \downarrow & & \downarrow \\ & B(G/T) & \xrightarrow{\quad} & B(G/RT) & \\ \swarrow & & \searrow & & \\ B(G/ST) & \xrightarrow{\quad} & X & & \end{array}$$

There is a natural isomorphism of groups (see [7]):

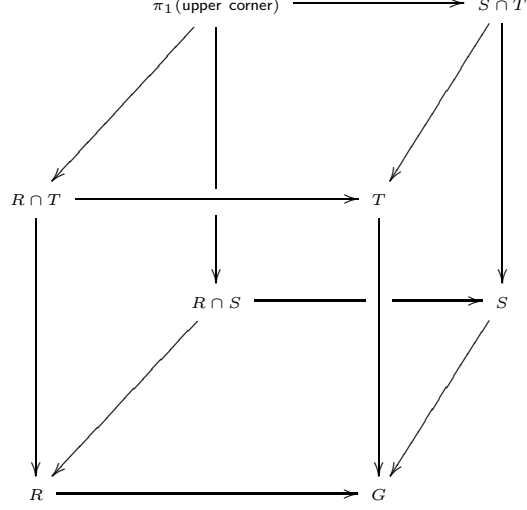
$$\pi_1(X) \simeq G/RST.$$

Certain higher homotopy groups of  $X$  are described in [7]. In particular, there is a natural isomorphism of  $\pi_1(X)$ -modules:

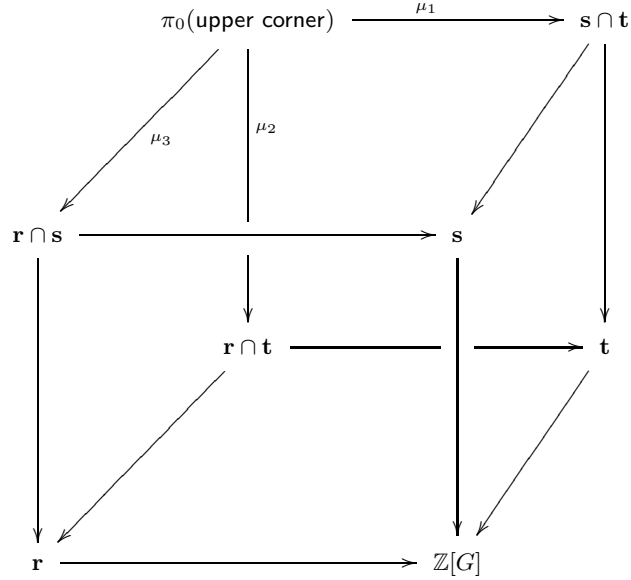
$$(17) \quad \pi_3(X) \simeq \frac{R \cap S \cap T}{\|R, S, T\|}$$

where the action of  $\pi_1(X) \simeq G/RST$  on the right hand side of (17) is viewed via conjugation in  $G$ . Recall the idea of the proof from [7]. Extend the above pushout to the cube of fibrations which have 27 spaces. The  $\pi_1$  of the complement to the

pushout in the cube of fibrations



is a crossed cube of groups. Moreover, it is a *universal* crossed cube of groups (see [7] for definition and discussion of the universality). One can realize the pushout diagram as a diagram of simplicial groups. The functor of group rings  $\mathbb{Z}[-] : \mathbf{groups} \rightarrow \mathbf{group\ rings}$  sends pushouts to the pushouts in the category of simplicial rings. Extending this pushout diagram to a cube of homotopy fibration sequences in the category of simplicial rings and taking  $\pi_0$  of the complement part as in (15) we obtain the crossed cube of rings



Now we observe that

$$\|\mathbf{r}, \mathbf{s}, \mathbf{t}\| \subseteq \text{Im}(\mu_i), \quad i = 1, 2, 3.$$

This follows from the properties of crossed cubes

$$\mu_i h(a \otimes b) = h(\mu_i a \otimes b) = h(a \otimes \mu_i b), \quad i = 1, 2, 3$$

for  $a \in \mathbf{r} \cap \mathbf{s}, b \in \mathbf{t}$  and other choices of ideals. Applying homotopy exact sequences of fibrations three times, and comparing them for simplicial groups and group rings, we obtain the needed commutative diagram

$$\begin{array}{ccc} \frac{R \cap S \cap T}{\|R, S, T\|} & \longrightarrow & \frac{\mathbf{r} \cap \mathbf{s} \cap \mathbf{r}}{\|\mathbf{r}, \mathbf{s}, \mathbf{t}\|} \\ \parallel & & \downarrow \\ \pi_2(\Omega X) & \xrightarrow{h_2 \Omega} & H_2(\Omega X) \end{array}$$

which considered together with proposition 4 imply the needed statement. Theorem 1 follows.  $\square$

**4.2. The case of two subgroups.** Observe that, in the case of two normal subgroups  $R, S$  in  $G$  is much simpler than the above case. In this case, one has a square of fibrations (in the category of simplicial rings)

$$\begin{array}{ccccc} T & \longrightarrow & \mathbf{r} & \longrightarrow & fib_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{s} & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G/S] \\ \downarrow & & \downarrow & & \downarrow \\ fib_1 & \longrightarrow & \mathbb{Z}[G/R] & \longrightarrow & \mathbb{Z}[X] \end{array}$$

such that

$$\begin{array}{ccc} \pi_0(T) & \xrightarrow{\mu_2} & \mathbf{r} \\ \downarrow \mu_1 & & \downarrow \\ \mathbf{s} & \longrightarrow & \mathbb{Z}[G] \end{array}$$

is a crossed square of rings. Since  $\mu_1 h(a \otimes b) = ab$ ,  $a \in \mathbf{s}, b \in \mathbf{r}$ , and  $\mu_1 h(a \otimes b) = ab$ ,  $a \in \mathbf{s}, b \in \mathbf{r}$ ,  $\mathbf{rs} + \mathbf{sr} \subseteq \text{Im}(\mu_1)$  and  $\mathbf{rs} + \mathbf{sr} \subseteq \text{Im}(\mu_2)$ . Comparing the picture for groups and group rings we conclude that there is a commutative diagram

$$\begin{array}{ccc} \frac{R \cap S}{[R, S]} & \xrightarrow{\quad} & \frac{\mathbf{r} \cap \mathbf{s}}{\mathbf{rs} + \mathbf{sr}} \\ \parallel & & \downarrow \\ \pi_1(\Omega X) & \xrightarrow{h_1 \Omega} & H_1(\Omega X) \end{array}$$

Since, for any connected space  $X$ , the Hurewicz homomorphism  $\pi_1(\Omega X) \rightarrow H_1(\Omega X)$  is a monomorphism, we obtain the following identification of the generalized dimension subgroup

$$D(G, \mathbf{rs} + \mathbf{sr}) = [R, S].$$

This gives a new proof of the result from [2]. This result can be generalized as follows.

Let  $T$  be a normal subgroup of subgroup of  $G$  and  $\mathbf{n} = (T - 1)\mathbb{Z}[G]$ . We obtain the following diagram

$$\begin{array}{ccc} \frac{R \cap S}{[R, S][R \cap S, T]} & \longrightarrow & \frac{\mathbf{r} \cap \mathbf{s}}{\mathbf{rs} + \mathbf{sr} + (\mathbf{r} \cap \mathbf{s})\mathbf{t} + \mathbf{t}(\mathbf{r} \cap \mathbf{s})} \\ \parallel & & \downarrow \\ \pi_1(\Omega X)_{TRS/RS} & \xrightarrow{(h_1\Omega)_{TRS/RS}} & H_1(\Omega X)_{TRS/RS}. \end{array}$$

Here the group  $TRS/RS$  is considered as a subgroup of  $\pi_1(X) = G/RS$ . We show that  $(h_1\Omega)_{TRS/RS}: \pi_1(\Omega X)_{TRS/RS} \rightarrow H_1(\Omega X)_{TRS/RS}$  is a monomorphism for any normal subgroup  $T$  of  $G$ . Let  $G_*$  be a simplicial group such that the geometric realization  $|G_*|$  is weakly homotopy equivalent to  $\Omega X$ . Observe that the action of  $\pi_1(X) \cong \pi_0(G_*)$  is induced by the conjugation action of  $G_0$  on  $G_*$ . Let  $\tilde{G}_*$  be the path-connected component of  $G_*$  containing the identity element. From the simplicial Postnikov system, there is a short exact sequence of simplicial groups

$$1 \longrightarrow \tilde{G}_* \longrightarrow G_* \longrightarrow \pi_0(G) \longrightarrow 1,$$

where  $\pi_0(G)$  is the discrete simplicial group. Hence

$$G_* = \coprod_{g \in \pi_0(G)} g\tilde{G}_*$$

as a simplicial set. Let  $\chi_h: G_* \rightarrow G_*, x \mapsto h x h^{-1}$  be the conjugation action of  $h \in G_0$  on  $G_*$ . Then  $\chi_h(g\tilde{G}_*) = h g h^{-1}\tilde{G}_*$  with

$$\chi_h(gx) = (h g h^{-1})(h x h^{-1}).$$

This implies that

$$H_k(\Omega X) \cong H_*(G_*) \cong H_k(\tilde{G}_*) \otimes \mathbb{Z}[\pi_0(G_*)]$$

as modules over  $\mathbb{Z}[\pi_0(G_*)]$  for  $k \geq 0$ , where  $\mathbb{Z}[\pi_0(G_*)]$  acts diagonally on the tensor product  $H_k(\tilde{G}_*) \otimes \mathbb{Z}[\pi_0(G_*)]$ . It follows that  $H_k(\Omega_0 X) \cong H_k(\tilde{G}_*)$  is a  $\mathbb{Z}[\pi_0(G_*)] \cong \mathbb{Z}[\pi_1(X)]$ -equivariant summand of  $H_k(\Omega X)$ , where  $\Omega_0 X$  is the path-connected component of  $X$  containing the basepoint. By taking  $k = 1$  with using the fact that  $\pi_1(\Omega X) \cong H_1(\Omega_0 X)$ , we have  $(h_1\Omega)_{TRS/RS}: \pi_1(\Omega X)_{TRS/RS} \rightarrow H_1(\Omega X)_{TRS/RS}$  is a monomorphism for any normal subgroup  $T$  of  $G$ . As a consequence, we obtain that

$$(18) \quad \frac{R \cap S}{[R, S][R \cap S, T]} \longrightarrow \frac{\mathbf{r} \cap \mathbf{s}}{\mathbf{rs} + \mathbf{sr} + (\mathbf{r} \cap \mathbf{s})\mathbf{t} + \mathbf{t}(\mathbf{r} \cap \mathbf{s})}$$

is a monomorphism for any normal subgroup  $T \leq G$ . We proved the following

**Theorem 9.** *For a group  $G$  and its normal subgroups  $R, S, T$ ,*

$$D(G, \mathbf{rs} + \mathbf{sr} + (\mathbf{r} \cap \mathbf{s})\mathbf{t} + \mathbf{t}(\mathbf{r} \cap \mathbf{s})) = [R, S][R \cap S, T].$$

## 5. GENERALIZATIONS AND EXAMPLES

**5.1. Simplicial groups.** Let  $G$  be a group and  $R_1, \dots, R_n$ ,  $n \geq 2$  its normal subgroups. Denote

$$\|R_1, \dots, R_n\| := \prod_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]$$

Similarly, for a collection  $\mathbf{a}_0, \dots, \mathbf{a}_n$  of ideals in a ring  $R$ , denote

$$\|\mathbf{a}_1, \dots, \mathbf{a}_n\| := \sum_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} (\cap_{i \in I} \mathbf{a}_i)(\cap_{j \in J} \mathbf{a}_j) + (\cap_{j \in J} \mathbf{a}_j)(\cap_{i \in I} \mathbf{a}_i).$$

It is easy to check that for arbitrary ideals  $\mathbf{a}, \mathbf{b}$  of  $\mathbb{Z}[G]$  we have  $D(\mathbf{a})D(\mathbf{b}) \subseteq D(\mathbf{a} + \mathbf{b})$  and  $[D(\mathbf{a}), D(\mathbf{b})] \subseteq D(\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a})$ . Indeed, the first inclusion is obvious, and the second follows from the equality  $g^{-1}h^{-1}gh - 1 = g^{-1}h^{-1}((g-1)(h-1) - (h-1)(g-1))$  for arbitrary  $g, h \in G$ . Therefore, for any  $G$  and its normal subgroups  $R_1, \dots, R_n$ ,

$$\|R_1, \dots, R_n\| \subseteq D(G, \|\mathbf{r}_1, \dots, \mathbf{r}_n\|),$$

where  $\mathbf{r}_i = (R_i - 1)\mathbb{Z}[G]$ . One can ask how to identify the generalized dimension quotient in the case of  $> 3$  subgroups or at least to describe its exponent. Here is an approach for constructing of different examples.

If  $\mathcal{G}$  is a simplicial group, we denote by  $N\mathcal{G}$  its Moore complex. Cycles of this complex we denote by  $Z_n\mathcal{G} = \text{Ker}(d : N_n\mathcal{G} \rightarrow N_{n-1}\mathcal{G})$ , and boundaries by  $B_n\mathcal{G} = \text{Im}(d : N_{n+1}\mathcal{G} \rightarrow N_n\mathcal{G})$ . Then  $\pi_n\mathcal{G} = Z_n\mathcal{G}/B_n\mathcal{G}$ . The following theorem is Theorem B of [4].

**Theorem 10** ([4]). *Let  $\mathcal{G}$  be a simplicial group and  $n \geq 1$  such that  $\mathcal{G}_{n+1}$  is generated by degeneracies. Set  $K_i := \text{Ker}(d_i : \mathcal{G}_n \rightarrow \mathcal{G}_{n-1})$ . Then*

$$B_n\mathcal{G} = \|K_0, \dots, K_n\|, \quad \pi_n\mathcal{G} = \frac{\bigcap_{i=0}^n K_i}{\|K_0, \dots, K_n\|}.$$

The following theorem is an analogue of the previous one for the case of simplicial rings. It follows from Theorem A of [4].

**Theorem 11** ([4]). *Let  $\mathcal{R}$  be a simplicial ring and  $n \geq 1$  such that  $\mathcal{R}_{n+1}$  is generated by degeneracies as a ring. Set  $\mathbf{k}_i := \text{Ker}(d_i : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1})$ . Then*

$$B_n\mathcal{R} = \|\mathbf{k}_0, \dots, \mathbf{k}_n\|, \quad \pi_n\mathcal{R} = \frac{\bigcap_{i=0}^n \mathbf{k}_i}{\|\mathbf{k}_0, \dots, \mathbf{k}_n\|}.$$

For a simplicial group  $\mathcal{G}$ , the Hurewicz homomorphism  $h : \pi_n\mathcal{G} \rightarrow H_n\mathcal{G}$  for  $n \geq 1$  is induced by the map  $\mathcal{G} \rightarrow \mathbb{Z}[\mathcal{G}]$  given by  $g \mapsto g - 1$ . The following statement is a direct corollary of theorems 10 and 11.

**Proposition 12.** *Let  $\mathcal{G}$  be a simplicial group and  $n \geq 1$  such that  $\mathcal{G}_{n+1}$  is generated by degeneracies. Set*

$$\begin{aligned} K_i &:= \text{Ker}(d_i : \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}) \\ \mathbf{k}_i &:= \text{Ker}(d_i : \mathbb{Z}[\mathcal{G}_n] \rightarrow \mathbb{Z}[\mathcal{G}_{n-1}]). \end{aligned}$$

Then

$$\frac{D(\mathcal{G}_n, \|\mathbf{k}_0, \dots, \mathbf{k}_n\|)}{\|K_0, \dots, K_n\|} = \text{Ker}(h_n : \pi_n\mathcal{G} \rightarrow H_n\mathcal{G}),$$

where  $h_n$  is the  $n$ th Hurewicz homomorphism.

The generalized dimension subgroups as in proposition 12 were considered in [15] for the case of simplicial Carlsson's constructions. The main example which we will consider here is the  $p$ -Moore space  $P^3(p) = S^2 \cup_p e^2$  for a prime  $p \geq 2$ . The lowest homotopy group of  $P^3(p)$  which contains  $\mathbb{Z}/p^2$ -summand is  $\pi_{2p-1}P^3(p)$ . This was proved in [5] for  $p > 3$ , however,  $\pi_3P^3(2) = \mathbb{Z}/4$ ,  $\pi_5P^3(3) = \mathbb{Z}/9$ . Since all homology groups  $H_*(\Omega P^3(p))$  have exponent  $p$ , we have

$$\mathbb{Z}/p \subseteq \text{Ker}\{h_{2p-2} : \pi_{2p-2}(\Omega P^3(p)) \rightarrow H_{2p-2}(\Omega P^3(p))\}.$$

Taking  $\mathcal{G}$  to be a simplicial model for  $\Omega P^3(p)$  with  $\mathcal{G}_3$  generated by degeneracies, we obtain the following example. Set  $G = \mathcal{G}_{2p-2}$ ,  $K_i = \text{Ker} d_i : \mathcal{G}_{2p-2} \rightarrow \mathcal{G}_{2p-3}$ , then, by proposition 12, the generalized dimension quotient

$$\frac{D(G, \|\mathbf{k}_0, \dots, \mathbf{k}_{2p-2}\|)}{\|K_0, \dots, K_{2p-2}\|}$$

contains a subgroup  $\mathbb{Z}/p$ .

In the case  $p = 2$ , we can choose

$$\mathcal{G}_1 = F(\sigma), \quad \mathcal{G}_2 = F(a, b, c)$$

with the face maps

$$d_0 : \begin{cases} a \mapsto \sigma^2 \\ b \mapsto \sigma \\ c \mapsto 1 \end{cases}, \quad d_1 : \begin{cases} a \mapsto 1 \\ b \mapsto \sigma \\ c \mapsto \sigma \end{cases}, \quad d_2 : \begin{cases} a \mapsto 1 \\ b \mapsto 1 \\ c \mapsto \sigma \end{cases}$$

Taking the kernels  $R = \text{Ker}(d_0)$ ,  $S = \text{Ker}(d_1)$ ,  $T = \text{Ker}(d_2)$ , we obtain an example discussed in introduction, with

$$\frac{D(G, \|\mathbf{r}, \mathbf{s}, \mathbf{t}\|)}{\|R, S, T\|} \simeq \mathbb{Z}/2.$$

**Conjecture.** For a prime  $p$ , any group  $G$  and its normal subgroups  $R_1, \dots, R_n$ ,  $n \geq 2$ , the quotient

$$\frac{D(G, \|\mathbf{r}_1, \dots, \mathbf{r}_n\|)}{\|R_1, \dots, R_n\|}$$

is  $p$ -torsion free provided  $n < 2p - 1$ .

**5.2. Connectivity conditions.** Let  $G$  be a group with normal subgroups  $R_1, \dots, R_n$ ,  $n \geq 2$ . Recall the connectivity condition from [7]. The  $n$ -tuple of normal subgroups  $(R_1, \dots, R_n)$  is called *connected* if either  $n \geq 2$  or  $n \geq 3$  and for all subsets  $I, J \subseteq \{1, \dots, n\}$ , with  $|I| \geq 2$ ,  $|J| \geq 1$ , the following holds

$$\left( \bigcap_{i \in I} R_i \right) \left( \prod_{j \in J} R_j \right) = \bigcap_{i \in I} \left( R_i \left( \prod_{j \in J} R_j \right) \right)$$

Now consider the homotopy colimit  $X$  of classifying spaces  $B(G / \prod_{i \in I} R_i)$ , where  $I$  ranges over all proper subsets  $I \subset \{1, \dots, n\}$ . Then, if for any  $i = 1, \dots, n$ , the  $n - 1$ -tuple of normal subgroups  $(R_1, \dots, \hat{R}_i, \dots, R_n)$  is connected, then

$$\pi_n(X) = \frac{R_1 \cap \dots \cap R_n}{\|R_1, \dots, R_n\|}.$$

This is proved in [7]. The proof of theorem 1 from [7] together with results of section 3, namely, together with the construction of the functor  $\Pi$ , imply the following



**Proposition 13.** *If for any  $i = 1, \dots, n$ , the  $n - 1$ -tuple of normal subgroups  $(R_1, \dots, \hat{R}_i, \dots, R_n)$  is connected, then there is a commutative diagram*

$$\begin{array}{ccc} \frac{R_1 \cap \dots \cap R_n}{\|R_1, \dots, R_n\|} & \longrightarrow & \frac{\mathbf{r}_1 \cap \dots \cap \mathbf{r}_n}{\|\mathbf{r}_1, \dots, \mathbf{r}_n\|} \\ \parallel & & \downarrow \\ \pi_{n-1}(\Omega X) & \xrightarrow{h_{n-1}\Omega} & H_{n-1}(\Omega X) \end{array} .$$

One can use proposition 13 for proving the above conjecture about the  $p$ -torsion in the generalized dimension quotient in some particular cases.

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